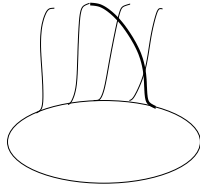


$\chi 1$: Algebraic Topology

- (1) Consider the knot $K \subset S^3$ depicted below. It is realized as a simple closed curve on a *standardly embedded* torus $T \subset S^3$, meaning that $S^3 \setminus T$ consists of two open solid tori.



- (a) Choose a basepoint $x_0 \in S^3 \setminus K$ and determine a presentation of $\pi_1(S^3 \setminus K; x_0)$.

x2: Differential Topology

- (1) If M is a smooth manifold, show that the tangent bundle TM and the cotangent bundle T^*M are isomorphic. (*Just as with vector spaces, there is no canonical isomorphism. You don't have to prove this, though. Also, feel free to assume anything that you like from linear algebra.*)
- (2) A *Lie homomorphism* is a smooth homomorphism between Lie groups.
- (a) Show that any Lie homomorphism $\phi : G \rightarrow H$ has constant rank: that is, there exists some $k \in \mathbb{Z}$ such that $\text{rank}(d\phi_g) = k$ for all $g \in G$.
- (b) Suppose that G, H are connected n -dimensional Lie groups and $\phi : G \rightarrow H$ is a Lie homomorphism with discrete kernel. Show that ϕ is a surjective diffeomorphism. (*In fact, ϕ is a covering map, but the proof of this is homework-level rather than exam-level.*)
- (3) Write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and let G be the pseudogroup generated by all diffeomorphisms ϕ between open subsets of \mathbb{R}^n that take horizontal factors to horizontal factors: that is,

$$\phi(x; y) = (\phi_1(x; y); \phi_2(y));$$

for $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Show that G consists of all diffeomorphisms between open subsets of \mathbb{R}^n whose Jacobian matrix at every point is an $n \times n$ matrix such that the lower left $(n-k) \times k$ block is 0. (*Showing that the set of diffeomorphisms satisfying the Jacobian property is a pseudo-group is almost immediate, although you should at least say what the properties are. The real point here is to explain why it is the minimal pseudo-group containing all such ϕ .*)

A G -structure on an n -manifold M is called a *codimension k foliation* of M . Since at least locally, the transition maps preserve the decomposition of \mathbb{R}^n into horizontal slices, these slices piece together to give a decomposition of M into submanifolds, called the *leaves* of the foliation.

- (4) Show that the antipodal map $A : S^n \rightarrow S^n$, $A(x) = -x$ is homotopic to the identity if and only if n is odd. (*Feel free to use Lefschetz theory if you would like.*)
- (5) Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^* \omega = 0$ for every smooth map $f : S^1 \rightarrow M$. (*Feel free to use Stokes' theorem, but you shouldn't reference deRham cohomology or anything that implicitly relies on this result.*)