ANALYSIS QUALIFYING EXAM

SEPTEMBER , 2012

REAL ANALYSIS

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Question 1 (30 points)

Let (X; M;) be a measure space. A measure M , with (E) = 1 , there

2 M such that 0 < (F) < 1 .

is semi nite and (E) = 1 , for any C > 0 there exists anF 2 M such that C < (F) <

(20 points)

) be a measure space and L p(X) = f f : X ! C : f is measurable and jjf jj p < 1g .

p(X) is a Banach space for 1 p < 1 by proving

p(X) then jjf + gjj p jj f jj p + jjgjj p

complete.

(30 points)

riation of a complex measure is the positive measure j j determined by the property

fd for some positive measure , f 2 L 1( ), then dj j = jf jd .

is is well de ned by showing the following;

ays exists such a measure.

tion is independent of .

(20 points)

:jj 2 be two norms on a vector space V such that jjvjj 1 jj vjj 2 for all v 2 V. If V is complete

to both norms, prove that they are equivalent.

be Banach spaces and let T n 2 L(X; Y) such that T(x) = lim n!1 T n(x) exists for all x 2 X.

2 L(X; Y).

COMPLEX ANALYSIS

You should attempt all the problems. Partial credit will be give for serious efforts

(1) Compute the following integral:

$$\int_0^\infty \frac{\log}{^2 + 1}$$

(2) Let $\mathbb{A} = \{ \ _0 \ \ _1 \ \ \ _n\}$ be a finite set of (distinct) points in the unit disk D. Define

$$_{\mathbb{A}}(\ ) = \Pi^n_{=0} \frac{-}{1 - -} \frac{| \ |}{} \qquad \text{for} \quad D$$

where if $= 0$, we set $\frac{| \ _i|}{_i} = 1$.

(a) Prove that $(\ )$ maps D to D and maps the unit circle to the unit circle.

(b) Let $T : D \to D$ be a fractional linear transformation that maps the unit disk onto itself. Prove that

$$_{\mathbb{A}} \circ T = \quad _{-1(\mathbb{A})}$$

where is a constant with $| \ | = 1$ and $T^{-1}(\mathbb{A}) = \{T^{-1}(\ _0) \quad T^{-1}(\ _n)\}$.

(c) Let $: D \to D$ be an analytic function with $(\ ) = 0$ for each $\mathbb{A}$. Prove that $| \ (\ )| \leq | \ _{\mathbb{A}}(\ )|$ for all D.

(3) The expression

$$\{ \quad \} = \frac{'''(\ )}{'(\ )} - \frac{3}{2}\left(\frac{''(\ )}{'(\ )}\right)^2$$

is called the c zn o . If $(\ )$ has a zero or pole of order $(\quad 1)$ at $_0$, show that $\{ \quad \}$ has a pole at $_0$ of order 2 and calculate the coe cient of $\frac{1}{(- \ _0)^2}$ in the Laurent development of $\{ \quad \}$.

(4) Let be a bounded

(a) Show that the area integral

$$\iint_{|\ \ |<1} \frac{(\ \ )\,\mathrm{d}x\,\mathrm{d}}{(1-\bar{\ }\ \ )^2} \qquad = x + i$$

is equal to

$$\int_0^1 \left(\int_{|\ \ |=1} \frac{(\ \ \ )}{i(\ \ -\ \ \ )^2}\,\mathrm{d} \right) \ \mathrm{d}$$

(Hint: use polar coordinates)

(b) Use part (a) to prove

$$(\ \ ) = \frac{1}{\ } \iint_{|\ \ |<1} \frac{(\ \ )\,\mathrm{d}x\,\mathrm{d}}{(1-\bar{\ }\ \ )^2}$$