Algebra Qualifying Exam

Fall 2014

You have 3 hours to answer all questions.

- 1. Determine the number of conjugacy classes of elements of order 4 in $GL_4(\mathbb{C})$, and give a representative of each class. Do the same for $GL_4(\mathbb{F}_7)$.
- **2.** Find the Galois groups of $f(x) = x^5 + 7x^3 + 6x^2 + x + 5$ over F_2 , F_3 , F_5 , and Q. You may assume without proof that $f(x) \supseteq F_3[x]$ has no irreducible quadratic factors.
- 3. Let R be a Noetherian ring. For any ideal J R de ne

$$D_{\overline{J}} = fx 2R : x^k 2J$$
 for some $k 2Z^+g$:

If ${}^{D}\overline{J} = J$, show that J can be expressed as a nite intersection of prime ideals. Hint: among all counterexamples, a maximal one cannot be prime.

- 4. Classify the groups of order 2915 = 5 11 53.
- **5.** Suppose R is a PID and A and B are R-modules. Let B_{tors} B be the submodule of R-torsion elements. Prove that

$$\operatorname{Tor}_{1}^{R}(A;B) = \operatorname{Tor}_{1}^{R}(A;B_{\operatorname{tors}})$$
:

- **6.** Suppose R is a Noetherian ring and \mathfrak{p} R is a prime ideal. Show that there is an $r \otimes \mathfrak{p}$ such that $S^{-1}R / R_{\mathfrak{p}}$ is injective, where S = f1/r; $r^2/r^3/\ldots g$.
- 7. Suppose R is a commutative local ring, and M and N are R-modules satisfying

$$M R N = 0$$
:

- (a) If M and N are nitely generated, show that either M=0 or N=0.
- (b) Show by example that (a) is false if we drop the hypothesis that M and N are nitely generated.
- **8.** Let $_8$ $_2$ C be a primitive eighth root of unity. The ring of integers in Q[$_8$] is Z[$_8$] (you may assume this without proof). If p is a prime, determine the number of primes of Z[$_8$] above p when
 - (a) p = 2,
 - (b) p 1 (mod 8),
 - (c) $p = 3.5.7 \pmod{8}$.